p-arton model for modular cusp forms

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Outline

1. Introduction
   - Modular Cusp Forms and L functions
   - p-adic wavelets
2. Vectors for modular cusp forms
   - Dual of the q-expansion
   - Melin transform of the p-arton
   - Hecke operators
   - Products of Dirichlet L-functions

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Modular Cusp forms

- A modular form $f : \mathbb{H} \to \mathbb{C}$ of weight k and level N associated to a Dirichlet character χ_N modulo N, which is holomorphic on the upper half plane $\mathbb{H}$ and transforms under the action of $\Gamma(N)$ a discrete subgroup of $SL(2, \mathbb{Z})$

$$f(\gamma z) = \chi_N(d)(cz + d)^k f(z) \tag{1}$$

- Using $z \to z + 1$ above, one sees that a modular form f of the full modular group has the following Fourier expansion in terms of $q = e^{2\pi iz}$:

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \tag{2}$$

- A cusp form is a modular form that vanishes at $Im(z) \to \infty$ or equivalently $q \to 0$. Implies $a(0) = 0$.

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Dirichlet series of a Cusp form

- A Dirichlet series of a cusp form is defined by the coefficients in its q-expansion :

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{3}$$

- This can be obtained by the Mellin transform of the cusp form :

$$L(s,f) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} dy\, y^{s-1} f(iy) \tag{4}$$

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$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} \tag{5}$$

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- It exhibits a q expansion :

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n \tag{6}$$

- The coefficients $\tau(n)$ satisfy the following properties :

$$\tau(m)\tau(n) = \tau(mn) \text{ if gcd } (m, n) = 1 \tag{7}$$

$$\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1}) \; m > 0 \tag{8}$$

$$|\tau(p)| \leq 2p^{\frac{11}{2}} \tag{9}$$

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- Using these properties, $L(s,f)$ exhibits an Euler product :

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \in primes} (1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s})^{-1} \quad (11)$$

- Note the quadratic form in the denominator.
- The motivation of our work is to find a dual description of the q-expansion via p-adic wavelets, which we shall call p-artons, and associate the Euler factor of the corresponding L-function with the Mellin transform of the p-artons.
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- The analog of the Haar wavelets on R are the Kozyrev wavelets on $\mathbb{Q}_p$:

$$\psi^{(p)}_{n,m,j}(x) = p^{-\frac{n}{2}} \exp[\frac{2\pi i}{p} j p^n x]\Omega_p(p^n x - m) \quad (12)$$

$$\int_{\mathbb{Q}_p} \psi^{(p)}_{n,m,j}(x)\psi^{(p)}_{n',m',j'}(x) = \delta_{nn'}\delta_{mm'}\delta_{jj'} \quad (13)$$

- where $n \in \mathbb{Z}$, $m \in \mathbb{Q}_p/\mathbb{Z}_p$ and $j = 1, 2, \ldots, p-1$ and

$$\Omega_p(x, x_0) = \begin{cases} 1 & \text{if } |x - x_0|_p \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad x, x_0 \in \mathbb{Q}_p \quad (14)$$

- The above can be obtained from the mother wavelets $\psi^{(p)}_{0,0,j}(x)$ by action of the affine group just like in the real case.

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- These are eigenfunctions of the Vladimirov derivative, defined as:

$$D^{\alpha}_{(p)} f(x) = \frac{1}{\Gamma_{(p)}(-\alpha)} \int dx' \frac{f(x') - f(x)}{|x - x'|_p^{\alpha+1}} \tag{15}$$

$$\Gamma_{(p)}(-\alpha) = \int_{\mathbb{Q}_p^{\times}} \frac{dx}{|x|_p} e^{2\pi i x} |x|_p^{-\alpha} \tag{16}$$

$$D^{\alpha}_{(p)} \psi^{(p)}_{n,m,j}(x) = p^{\alpha(1-n)} \psi^{(p)}_{n,m,j}(x) \tag{17}$$

- Our interest is only on the index n related to the scaling. We will use the ket-notation associated with the wavelet when ever possible:

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- We can define raising and lowering operators $a_{\pm}^{(p)}$ on the wavelets that changes the scaling quantum number by one:

$$a_{\pm}^{(p)}\psi_{n,0,1}^{(p)}(x) = \psi_{n\pm1,0,1}^{(p)}(x) \longleftrightarrow a_{\pm}^{(p)}|n\rangle_{(p)} = |n \mp 1\rangle_{(p)} \qquad (19)$$

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- The wavelets supported on this subset, correspond to the eigen values $\{1, p, p^2, \ldots\}$ of the operator $D_{(p)}$
- When restricted to this subspace, we demand that the wavelet $\psi_{1,0,1}^{(p)}(x)$ is the ground state:

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Dual of the q-expansion

- We use the prime factorization of a natural number $n \in \mathbb{N}$ to relate it to a wavelet in $\otimes \mathcal{H}_-^{(p)} \subset \otimes_p \mathbb{Q}_p$:

$$\begin{aligned} n &= \prod_p p^{n_p} \longmapsto \\ \bigotimes_p |n_p\rangle_{(p)} &= |n_2\rangle_{(2)} \otimes |n_3\rangle_{(3)} \otimes |n_5\rangle_{(5)} \otimes |n_7\rangle_{(7)} \otimes \cdots \end{aligned} \tag{21}$$

- Here all but a finite number of n_p are zero.
- Next we associate with a cusp form a vector in $\otimes \mathcal{H}_-^{(p)}$:

$$\begin{aligned} f = \sum_{n=1}^{\infty} a(n) q^n &= \sum_{n_p=0}^{\infty} \left(\prod_p a(p^{n_p}) \right) q^{\prod p^{n_p}} \\ \longmapsto \quad |\mathfrak{f}\rangle &= \sum_{n_p=0}^{\infty} \bigotimes_p a(p^{n_p}) \, |n_p\rangle_{(p)} \end{aligned} \tag{22}$$

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- Which can be simplified using the multiplicative property of the coefficients, and by action of the lowering operator on the ground state: $|n_p\rangle = a_-^{n_p}|0\rangle_{(p)}$

$$= \sum_{\substack{n_2, n_3, n_5, \\ \cdots = 0}}^{\infty} a(2^{n_2})|n_2\rangle_{(2)} \otimes a(3^{n_3})|n_3\rangle_{(3)} \otimes a(5^{n_5})|n_5\rangle_{(5)} \otimes \cdots$$
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- We consider the vector $\left|\mathfrak{f}_{(p)}\right\rangle \in \mathcal{H}_{-}^{(p)}$ as the p-th p-arton, i.e. 'part' of the cusp form f at the prime p.
- The operator acting on the ground state resembles the form of the Euler factor for prime p of the L- function associated with f.
- Explicitly, for each $\left|\mathfrak{f}_{(p)}\right\rangle$ we have a $f_{(p)}(x_{(p)})$ supported on $p^{-1}\mathbb{Z}_p$:

$$f_{(p)}(x_{(p)}) = \sum_{n_p=0}^{\infty} a(p^{n_p})\psi^{(p)}_{1-n_p,0,1}(x_{(p)}) \tag{25}$$

- In summary, according to this correspondence, a cusp form $f : \mathbb{H} \to \mathbb{C}$ is equivalent to the infinite set of functions $f_{(p)} : \mathbb{Q}_p \to \mathbb{C}$, one function for each prime p. The two are equivalent in the sense that from f we can get $\left(f_{(2)}, f_{(3)}, f_{(5)}, \cdots\right)$ and vice versa.

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- In summary, according to this correspondence, a cusp form $f : \mathbb{H} \to \mathbb{C}$ is equivalent to the infinite set of functions $f_{(p)} : \mathbb{Q}_p \to \mathbb{C}$, one function for each prime p. The two are equivalent in the sense that from f we can get $\left(f_{(2)}, f_{(3)}, f_{(5)}, \cdots\right)$ and vice versa.

- We consider the vector $\left|\mathfrak{f}_{(p)}\right\rangle \in \mathcal{H}_{-}^{(p)}$ as the p-th p-arton, i.e. 'part' of the cusp form f at the prime p.
- The operator acting on the ground state resembles the form of the Euler factor for prime p of the L- function associated with f.
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Melin transform of the p-arton

- Consider the Mellin transform defined using the unitary character $\omega_\ell = e^{\frac{2\pi i \ell}{p} x|x|_p}$ which determines a phase depending on the leading $p-$adic 'digit' of x :

$$\tilde{g}_\omega(s) \equiv \mathcal{M}_{(p,\omega)}[g](s) = \int_{\mathbb{Q}_p^\times} d^\times x \, e^{\frac{2\pi i \ell}{p} x|x|_p} \, |x|_p^s \, g(x),$$
$$s \in \mathbb{C} \text{ and } \ell = 0, 1, \cdots, p-1 \qquad (26)$$

- Using the above the Mellin transform of the Kozyrev wavelet $\psi_{n,0,1}(x)$ can be evaluated :

$$\mathcal{M}_{(p,\omega)}[\psi_{n,1,0}](s) = -\left(\frac{1}{p(1-p^{-s})} - \frac{1}{p^s - 1}\delta_{l,0} - \delta_{l,p-1}\right) p^{n\left(s-\frac{1}{2}\right)} \qquad (27)$$

- Thus the Mellin transform of the p-arton yields:

$$\mathcal{M}_{(p,\omega)}\left[\mathfrak{f}_{(p)}(x_{(p)})\right](s) = c_p(\ell, s) \sum_{n_p=0}^{\infty} a(p^{n_p})\, p^{(1-n_p)\left(s-\frac{1}{2}\right)} \qquad (28)$$

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- We can combine the results for all the primes, the Mellin transform of the full wavefunction associated with the modular form is then (plugging in same argument for all the primes):

$$\begin{aligned} &\mathcal{M}_{(p,\omega)}\left[\left\langle\left(\xi_{(2)},\xi_{(3)},\xi_{(5)},\cdots\right)\middle|\mathfrak{f}\right\rangle\right](s) \\ &= \prod_p \mathcal{M}_{(p,\omega)}\left[\mathfrak{f}_{(p)}(\xi_{(p)})\right](s) \\ &= \prod_p c_p(\ell,s)\sum_{n_p=0}^{\infty} a(p^{n_p})p^{(1-n_p)\left(s-\frac{1}{2}\right)} \\ &= \left(\prod_p c_p(\ell,s)p^{s-\frac{1}{2}}\right) L\left(s-\frac{1}{2},f\right) \end{aligned} \tag{29}$$

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Hecke operators

- Next our goal would be to have some operational understanding of the Hecke operators over these vectors.
- Recall the Hecke operators $T(m)$, $m \in \mathbb{N}$, are a set of commuting operators with action on an eigen cusp-form :

$$T(m)f(z) = a(m)f(z) \tag{30}$$

- They satisfy the following algebraic identities :

$$T(m)T(n) = T(mn) \text{ for } m \nmid n \tag{31}$$
$$T(p)T(p^{\ell}) = T(p^{\ell+1}) + \chi(p)p^{k-1}T(p^{\ell-1}) \tag{32}$$

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$$\Psi^{(p)}_{n,m,j}(x) = |x|_p^{\frac{1}{2}}\,\psi^{(p)}_{n,m,j}(x) \tag{38}$$

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$$\begin{aligned} \mathbf{a}_+\mathbf{f}_{(p)}(x) &= \sum_{n_p=1}^{\infty} \int_{\mathbb{Q}_p^\times} d^\times y\, \Psi^{(p)}_{2-n_p,0,1}(x)\Psi^{(p)*}_{1-n_p,0,1}(y)\mathbf{f}_{(p)}(y) \\ \mathbf{a}_-\mathbf{f}_{(p)}(x) &= \sum_{n_p=0}^{\infty} \int_{\mathbb{Q}_p^\times} d^\times y\, \Psi^{(p)}_{-n_p,0,1}(x)\Psi^{(p)*}_{1-n_p,0,1}(y)\mathbf{f}_{(p)}(y) \end{aligned} \tag{41}$$

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- This also means $\chi^{*\frac{1}{2}}(p)a_{\mathbf{f}}(p)$ are real.
- From the inner product, the Parseval indentity can be deduced in terms of the Euler factor of the L function, via the Mellin transform:

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$$p^{-\frac{k-1}{2}} \left(a_{\mathbf{f}}(p) - \chi(p) a^*_{\mathbf{g}}(p)\right) \int_{\mathbb{Q}_p^\times} d^\times x \, \mathbf{g}^*_{(p)}(x) \mathbf{f}_{(p)}(x) = 0 \qquad (43)$$

- This also means $\chi^{*\frac{1}{2}}(p) a_{\mathbf{f}}(p)$ are real.
- From the inner product, the Parseval indentity can be deduced in terms of the Euler factor of the L function, via the Mellin transform:

$$\begin{aligned} &\left(\mathbf{f}_{(p)} \middle| \mathbf{g}_{(p)}\right) \qquad (44) \\ &= \frac{\ln p}{2\pi} \int_0^{\frac{2\pi}{\ln p}} dt \left(L_{\mathbf{f}(p)}\left(\frac{k-1}{2} + it\right)\right)^* \left(L_{\mathbf{g}(p)}\left(\frac{k-1}{2} + it\right)\right) \end{aligned}$$

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$$a(p) = 2\cos\frac{1}{2}(\alpha_1 + \alpha_2)\, p^{\frac{k-1}{2}}\, e^{\frac{i}{2}(\alpha_1 - \alpha_2)}$$
$$\chi(p) = e^{i(\alpha_1(p) - \alpha_2(p))} \tag{45}$$

- Which is consistent with the condition on the growth of the coefficients $a(p)$.
- The function $L_p(s, f)$ is the generating function of the orthogonal Chebyshev polynomial of type **II**, denoted by $U_n(x)$, with $\theta = \frac{1}{2}(\alpha_1 + \alpha_2)$ and $t = p^{\frac{k-1}{2} - s} e^{\frac{i}{2}(\alpha_1 - \alpha_2)}$.

$$\frac{1}{1 - 2t\cos\theta + t^2} = \sum_{n=0}^{\infty} U_n(\cos\theta) t^n \tag{46}$$
$$U_{n+1}(\xi) = 2\zeta U_n(\xi) - U_{n-1}(\xi), \quad U_0(\xi) = 1 \text{ and } U_1(\xi) = 2\xi$$

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$$\sum_{n=0}^{\infty} U_n(\cos\phi) U_n(\cos\phi') \propto \delta(\phi - \phi') \tag{47}$$

- This explicitly proves the orthogonality of the p-artons which are eigen functions of the Hecke operator defined.
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Outline

Products of Dirichlet L-functions

- We look at a simpler case of a Dirichlet L-function, corresponding to the Dirichlet character ν :

$$L(s,\nu) = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \prod_{p \in \text{primes}} \frac{1}{(1-\nu(p)p^{-s})} \tag{48}$$

- Our goal is to use it to mimic the properties of a modular L-function.

$$_2\mathrm{L}(s,\nu) = L(s,\nu)L(s,\nu^*) = \prod_p \frac{1}{(1-\nu(p)p^{-s})\,(1-\nu^*(p)p^{-s})}$$
$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \frac{1}{(1-2\cos(\arg \nu_p)p^{-s}+p^{-2s})} \tag{49}$$

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$$ {}_2\mathrm{L}_p(s,\nu) = \frac{1}{(1-2\cos(\arg\nu_p)p^{-s}+p^{-2s})} = \sum_{n_p=0}^{\infty} U_{n_p}(\cos(\arg\nu_p))p^{-sn_p} \tag{50} $$

- The Dirichlet L- function is related to the ϑ-series :

$$ \vartheta(z,\nu) = \sum_{n\in\mathbb{Z}} \nu(n) n^{\epsilon} e^{i\pi n^2 z/N} \tag{51} $$

- Here ν is a primitive Dirichlet character modulo N and $\epsilon = \frac{1}{2}(1-\nu(-1))$ takes the value 0 or 1 depending on whether ν is even or odd, respectively. The Mellin transform of the above is the L-function :

$$ L(s,\nu) = \frac{(\pi/N)^{\frac{s+\epsilon}{2}}}{2\Gamma\left(\frac{s+\epsilon}{2}\right)} \int_0^{\infty} \frac{dy}{y} y^{\frac{s+\epsilon}{2}} \vartheta(iy,\nu) \tag{52} $$

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&{}_2\mathrm{L}(s,\nu) \\
&= \frac{(\pi/N)^{s+\epsilon}}{4\left(\Gamma\left(\frac{s+\epsilon}{2}\right)\right)^2}\int_0^\infty dy_1\int_0^\infty dy_2\,(y_1y_2)^{\frac{s+\epsilon}{2}-1}\vartheta(iy_1,\nu)\vartheta(iy_2,\nu^*) \\
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$$\mathrm{M}_N(x+iy) \sim \sqrt{y}\,\mathbf{f}(x,y,\nu) = \sum_{n=1}^{\infty} a_n (ny)^{\epsilon} \sqrt{y}\, K_0\left(\frac{2\pi n}{N} y\right) e^{2\pi i n x/N} \quad (56)$$

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- The motivation to do this was to construct a p-artonic model for such a function by choosing the p-th factor:

$$\mathbf{f}_{(p)}(\nu, x) = \sum_{n_p=0}^{\infty} a(p^{n_p})\Psi^{(p)}_{1-n_p,0,1}(x) = \sum_{n_p=0}^{\infty} U_{n_p}(\cos(\arg \nu_p))\, \Psi^{(p)}_{1-n_p,0,1}(x)$$

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- An analogous inner product exists on the space of Maass-like wave forms in the rectangular region $\{-\frac{N}{2} < |x| \leq \frac{N}{2}, y > 0\}$:

$$\langle \mathbf{f}(\nu_{\mathbf{f}}) | \mathbf{g}(\nu_{\mathbf{g}}) \rangle$$
$$= \sum_{m,n=1}^{\infty} a_{\mathbf{f}}^*(m) a_{\mathbf{g}}(n) \int_{-N/2}^{N/2} dx\, e^{2\pi i(n-m)x/N} \int_0^{\infty} \frac{dy}{y} K_0\left(\frac{2\pi my}{N}\right) K_0\left(\frac{2\pi ny}{N}\right)$$

- Which yields the same sum over the coefficients $a_{\mathbf{f}}^*(n) a_{\mathbf{g}}(n)$
- The above inner products can be related to the inner product of the corresponding L-functions :

$$\langle \mathbf{f}(\nu_{\mathbf{f}}) | \mathbf{g}(\nu_{\mathbf{g}}) \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{\mathbf{f}}^*(n)\, a_{\mathbf{g}}(m) \delta_{mn} \tag{59}$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \sum_{n=1}^{\infty} \frac{a_{\mathbf{f}}^*(n)}{m^{\frac{1}{2}-it}} \sum_{n=1}^{\infty} \frac{a_{\mathbf{g}}(n)}{n^{\frac{1}{2}+it}}$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, {}_2\mathrm{L}_{\mathbf{f}}^* \left(\frac{1}{2} - it, \nu_{\mathbf{f}}\right) {}_2\mathrm{L}_{\mathbf{g}} \left(\frac{1}{2} + it, \nu_{\mathbf{g}}\right)$$

- An analogous inner product exists on the space of Maass-like wave forms in the rectangular region $\{-\frac{N}{2} < |x| \leq \frac{N}{2},\, y > 0\}$:

$$\left\langle \mathbf{f}(\nu_{\mathbf{f}}) \middle| \mathbf{g}(\nu_{\mathbf{g}}) \right\rangle$$
$$= \sum_{m,n=1}^{\infty} a_{\mathbf{f}}^{*}(m) a_{\mathbf{g}}(n) \int_{-N/2}^{N/2} dx\, e^{2\pi i(n-m)x/N} \int_{0}^{\infty} \frac{dy}{y} K_0\left(\frac{2\pi m y}{N}\right) K_0\left(\frac{2\pi n y}{N}\right)$$

- Which yields the same sum over the coefficients $a_{\mathbf{f}}^{*}(n) a_{\mathbf{g}}(n)$
- The above inner products can be related to the inner product of the corresponding L-functions :

$$\begin{aligned} \left\langle \mathbf{f}(\nu_{\mathbf{f}}) \middle| \mathbf{g}(\nu_{\mathbf{g}}) \right\rangle &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{\mathbf{f}}^{*}(n)\, a_{\mathbf{g}}(m) \delta_{mn} \qquad (59) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \sum_{n=1}^{\infty} \frac{a_{\mathbf{f}}^{*}(n)}{m^{\frac{1}{2}-it}} \sum_{n=1}^{\infty} \frac{a_{\mathbf{g}}(n)}{n^{\frac{1}{2}+it}} \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, {}_2\mathrm{L}_{\mathbf{f}}^{*}\left(\frac{1}{2} - it, \nu_{\mathbf{f}}\right) {}_2\mathrm{L}_{\mathbf{g}}\left(\frac{1}{2} + it, \nu_{\mathbf{g}}\right) \end{aligned}$$

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- The final expression, although although obtained via some formal manipulation, is a Perseval type identity, interpreted as an inner product $\left\langle {}_2\mathsf{L}_{\mathbf{f}} \middle| {}_2\mathsf{L}_{\mathbf{g}} \right\rangle$
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Outlook

- We leave the study of the action of the group $GL(2, \mathbb{Q}_p)$ on these p-artons to the future. This might have connection to the classical theory of Automorphic forms.
- The construction is also suggestive of a holographic correspondence, where the data of a function f in the upper half of the complex plane is related to $\otimes_p \mathbb{Q}_p$, which in turn is related to $\mathbb{R} = \partial \mathbb{H}$
- Possible interpretation of the ${}_2L(s, f)$ as the Mellin dual of the p-artons and hence to construct Hecke-'like' operators on them.

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